Solution to the inverse scattering problem for strongly fluctuating media using partially coherent light

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We investigate the inverse scattering problem for statistically homogeneous, isotropic random media under conditions of strong fluctuations of optical wavefields. We present a method for determining the spectral density of the dielectric constant fluctuations in such media from scattering of partially coherent light. The method may find applications to a wide class of turbulent media such as the turbulent atmosphere and certain turbulent plasmas where backscattering and depolarization effects are negligible. © 2002 Optical Society of America

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The inverse scattering problem is one of the most ubiquitous problems in science (see, for example, Refs. 1–6). In the optical context, the problem of reconstructing the dielectric susceptibility of a deterministic, weakly scattering medium was solved in Ref. 7 within the accuracy of the first Born approximation. The generalization of the technique of that reference to random media has met with difficulties, however, owing to the nonuniqueness of reconstruction even for weakly scattering media.[8]

To date, successful reconstruction schemes have been developed either for weakly scattering, statistically quasi-homogeneous random media[9–11] or for strongly scattering turbid media in the diffusion approximation.[12] However, neither of these approximations can be applied to turbulent media under strong fluctuation conditions (Ref. 13, Sect. 3.2). Under these conditions, the fluctuations of the phase and the amplitude of a wavefield are strong enough so that neither the first Born approximation nor the first Rytov approximation is valid. On the other hand, light scattering is highly anisotropic, which makes the use of the isotropic diffusion approximation inappropriate.

In this Letter we present a solution to the inverse scattering problem for statistically homogeneous, isotropic random media in the *strong* fluctuation regime. Our approach is based on the inversion of a *closed-form nonperturbative solution* to the direct scattering problem. We show that the use of partially coherent, quasi-homogeneous incident beams makes it possible to reconstruct the spatial spectrum of the dielectric constant fluctuations in such media from measurements of the radiant intensity of the scattered light, thereby avoiding measurements of field correlations. The new method is expected to be applicable to gaseous turbulent media as well as to turbulent plasma, where a direct calculation of the spatial spectrum of homogeneous isotropic turbulence from the hydrodynamic equations presents formidable difficulties.[14]

We begin by considering a beam, propagating around the z direction into the half-space $z > 0$, where there is a random medium. The random medium may be specified by the fluctuation of the dielectric constant $\epsilon(\boldsymbol{\rho}, z) = \langle \epsilon(\boldsymbol{\rho}, z)\rangle + \epsilon_1(\boldsymbol{\rho}, z)$ which is assumed to be a statistically homogeneous and isotropic random field. Suppose that the beam emerges from the random medium after it has traveled the distance L. In the absence of backscattering and depolarization effects,[15] the cross-spectral density function of the beam (Ref. 16, Sect. 2.4) at a pair at points $\boldsymbol{\rho}_1$ and $\boldsymbol{\rho}_2$ in the plane $z = L$ is related to the cross-spectral density of the incident beam at $z = 0$ by the expression (Ref. 13, Sect. 3.2)

$$W(\boldsymbol{\rho}, \mathbf{R}, L) = \int \mathrm{d}^2 R' \int \mathrm{d}^2 \rho' W_0(\mathbf{R}' - \boldsymbol{\rho}'/2, \mathbf{R}' + \boldsymbol{\rho}'/2) \times \left(\frac{k}{2\pi L}\right)^2 \exp\left[i\frac{k}{L}(\boldsymbol{\rho} - \boldsymbol{\rho}')\cdot(\mathbf{R} - \mathbf{R}')\right] \times \exp\left\{-\frac{\pi k^2}{4}\int_0^L \mathrm{d}z\, \mathcal{D}_\epsilon\left[\left|\frac{\boldsymbol{\rho} z}{L} + \boldsymbol{\rho}'\left(1 - \frac{z}{L}\right)\right|\right]\right\}\cdot \quad (1)$$

Here $W_0(\boldsymbol{\rho}_1, \boldsymbol{\rho}_2)$ is the cross-spectral density of the incident field, $k = \omega/c$, where ω is the frequency of the field, $\boldsymbol{\rho} = \boldsymbol{\rho}_1 - \boldsymbol{\rho}_2$ and $\mathbf{R} = (\boldsymbol{\rho}_1 + \boldsymbol{\rho}_2)/2$. Further, $\mathcal{D}_\epsilon(\rho)$ is the (suitably normalized) two-dimensional structure function of the random process $\epsilon_1(\boldsymbol{\rho}, z)$, which is related to the two-dimensional spectral density $\Phi_\epsilon(\mathrm{K}_\perp)$ by the formula (Ref. 13, Sect. 3.1)

$$\mathcal{D}_\epsilon(\rho) = 4\pi \int_0^\infty \mathrm{d}\mathrm{K}_\perp \mathrm{K}_\perp [1 - J_0(\mathrm{K}_\perp \rho)]\Phi_\epsilon(\mathrm{K}_\perp)\,. \quad (2)$$

In Eq. (2), the two-dimensional spectral density $\Phi_\epsilon(\mathrm{K}_\perp)$ is obtained from the three-dimensional one, $\Phi_\epsilon(\sqrt{\mathrm{K}_\perp^2 + \mathrm{K}_z^2})$, by letting $\mathrm{K}_z = 0$. Due to the complexity of expression (1), it is not suitable as a starting point for the determination of the structure function from scattering data. To solve this problem, we consider the case of a partially coherent incident beam

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that is generated by a planar, quasi-homogeneous source. The cross-spectral density of such a source has the form (Ref. 16, Sect. 5.3)

$$W_0(\mathbf{R} - \boldsymbol{\rho}/2, \mathbf{R} + \boldsymbol{\rho}/2) = I_0(\mathbf{R})\mu_0(\boldsymbol{\rho}), \tag{3}$$

where $I_0(\mathbf{R})$ and $\mu_0(\boldsymbol{\rho})$ are the intensity and the spectral degree of coherence of the field across the source, respectively. The effective width σ_I of the intensity distribution across such a source is much greater than the spectral coherence length σ_c of the source. On substituting from Eq. (3) into Eq. (1), one obtains, after straightforward algebra, the following expression for the cross-spectral density of the beam in the exit plane $z = L$:

$$W(\boldsymbol{\rho}, \mathbf{R}, L) = \left(\frac{k}{2\pi L}\right)^2 \int \mathrm{d}^2\boldsymbol{\rho}' \tilde{I}_0\left[\frac{k}{L}(\boldsymbol{\rho} - \boldsymbol{\rho}')\right]\mu_0(\boldsymbol{\rho}') \times \exp\left(\frac{ik}{L}\right)(\boldsymbol{\rho} - \boldsymbol{\rho}') \cdot \mathbf{R} \times \exp\left\{-\frac{\pi k^2}{4}\int_0^L \mathrm{d}z \mathcal{D}_\epsilon\left[\left|\frac{\boldsymbol{\rho} z}{L} + \boldsymbol{\rho}'\left(1 - \frac{z}{L}\right)\right|\right]\right\}. \tag{4}$$

Here $\tilde{I}_0(\mathbf{k})$ is a two-dimensional Fourier-transform of the intensity $I_0(\mathbf{x})$ in the source plane $z = 0$, defined as $\tilde{I}_0(\mathbf{k}) \equiv \int \mathrm{d}^2 x I_0(\mathbf{x})\exp(-i\mathbf{k} \cdot \mathbf{x})$.

The analysis of Eq. (4) indicates that the behavior of the integrand is determined by the relation among the spectral coherence length σ_c of the incident beam, the characteristic spatial scale $\Delta \sim L/k\sigma_I$, of the Fourier transform of the incident intensity, and a characteristic width of the exponential term at small spatial scales of the order of σ_c. It follows from Eq. (2) that, at small spatial scales, $\mathcal{D}_\epsilon(\rho) \sim \rho^2/l_D$, where $l_D^{-1} \sim \int_0^\infty \mathrm{dK}_\perp \mathrm{K}_\perp^3 \Phi_\epsilon(\mathrm{K}_\perp)$, which is of the order of a typical diffusion coefficient of the angular spread of the beam in the random medium.[17] The characteristic width of the exponential term can then be estimated as $\delta \sim (l_D/k^2L)^{1/2}$. It can be concluded from these considerations that $\tilde{I}_0(x)$ can be approximated by the Dirac delta function,

$$\tilde{I}_0\left[\frac{k}{L}(\boldsymbol{\rho} - \boldsymbol{\rho}')\right] \sim \left(\frac{L}{k}\right)^2 \delta(\boldsymbol{\rho} - \boldsymbol{\rho}'), \tag{5}$$

provided that

$$\frac{L}{k\sigma_I} \ll \min\left(\sigma_c, \sqrt{\frac{l_D}{k^2 L}}\right), \qquad \sigma_c \ll \sigma_I. \tag{6}$$

In physical terms, Eq. (5) means that in the paraxial approximation and under the condition given by Eq. (6), the field can be treated, to a good approximation, as a partially coherent plane wave. One can readily satisfy this condition in a laboratory. Consider, for example, a turbulent medium with $l_D \sim 10^{14}$ cm, typical of atmospheric turbulence,[17] and taking $L \sim 1$ m, $\sigma_I \sim 10$ cm, $k \sim 10^5$ cm^{-1}, one obtains $\sigma_c \gg 10^{-4}$ cm.

On combining Eqs. (4) and (5) and recalling the definition of the spectral degree of coherence (Ref. 16, Sect. 5.3.1),

$$\mu(\boldsymbol{\rho}_1, \boldsymbol{\rho}_2, z) \equiv \frac{W(\boldsymbol{\rho}_1, \boldsymbol{\rho}_2, z)}{\sqrt{W(\boldsymbol{\rho}_1, \boldsymbol{\rho}_1, z)}\sqrt{W(\boldsymbol{\rho}_2, \boldsymbol{\rho}_2, z)}}, \tag{7}$$

one obtains, in the present case, for the spectral degree of coherence of the light in the plane $z = L$ the expression

$$\mu(\boldsymbol{\rho}, L) = \mu_0(\boldsymbol{\rho})\exp\left[-\frac{\pi k^2}{4} L \mathcal{D}_\epsilon(\rho)\right], \tag{8}$$

which is a generalization of the corresponding result for a fully coherent plane wave (Ref. 13, Sect. 3.2). Equation (8) can be inverted at once, giving

$$\mathcal{D}_\epsilon(\rho) = \frac{4}{\pi k^2 L}\ln\left|\frac{\mu_0(\rho)}{\mu(\rho, L)}\right|. \tag{9}$$

Here we have assumed, for simplicity, that at any pair of points in the plane $z = 0$, the spectral degree of coherence of the incident beam depends only on the distance between the points, i.e., $\mu_0 = \mu_0(\rho)$. Although this assumption is not necessary for our purposes, the majority of partially coherent sources generated in a laboratory satisfies this condition.

Let us express the spectral density $\Phi_\epsilon(\mathrm{K}_\perp)$ in terms of the structure function $\mathcal{D}_\epsilon(\rho)$. For this purpose, we apply to both sides of Eq. (2) the two-dimensional Laplacian operator. Next, we multiply the resulting equation by $J_0(\mathrm{K}_\perp \rho)$ and use the Bessel equation as well as the orthogonality relation for the Bessel functions (Ref. 18, Sect. 2.6). We then obtain, after some algebra, the following expression for the two-dimensional spectral density:

$$\Phi_\epsilon(\mathrm{K}_\perp) = \frac{1}{4\pi \mathrm{K}_\perp^2}\int_0^\infty \mathrm{d}\rho J_0(\mathrm{K}_\perp \rho)\frac{\mathrm{d}}{\mathrm{d}\rho}\left(\rho\frac{\mathrm{d}\mathcal{D}_\epsilon}{\mathrm{d}\rho}\right). \tag{10}$$

Equations (9) and (10) are the main result of our analysis. They may be used to determine the structure function and the two-dimensional spectrum of the dielectric constant fluctuations from measurements of the spectral degree of coherence of the light beam that emerges from the random medium. It also readily follows from the assumed statistical isotropy of the medium that the three-dimensional spectral density $\Phi_\epsilon(\sqrt{\mathrm{K}_\perp^2 + \mathrm{K}_z^2})$ has the same functional form as the two-dimensional spectral density $\Phi_\epsilon(\mathrm{K}_\perp)$. Therefore, it can be inferred from Eqs. (9) and (10) that the three-dimensional spectral density of dielectric constant fluctuations is given by the expression

$$\Phi_\epsilon(\mathrm{K}) = \frac{1}{\pi^2 \mathrm{K}^2 k^2 L}\int_0^\infty \mathrm{d}\rho J_0(\mathrm{K}\rho) \times \frac{\partial}{\partial\rho}\left[\rho\frac{\partial \ln|\mu_0(\rho)/\mu(\rho, L)|}{\partial\rho}\right], \tag{11}$$

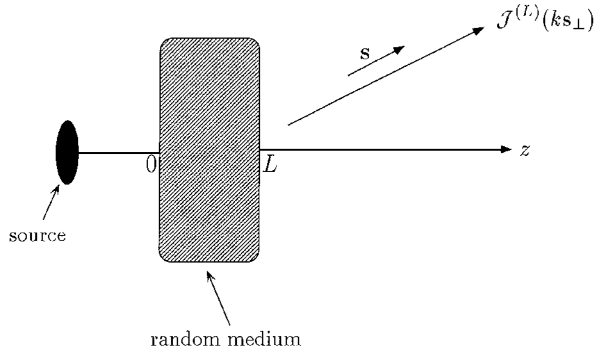

Fig. 1. Illustrating the notation.

where $K = \sqrt{K_\perp^2 + K_z^2}$. It is only necessary to measure the spectral degree of coherence of the incident and of the scattered beams.

To determine the spectral degree of coherence of the beam that has passed through a layer of thickness L in the turbulent medium, we recall that the incident beam was assumed to have been generated by a quasi-homogeneous source. Further, the spatial coherence of the beam can only decrease due to multiple scattering events on propagation in the layer of the turbulent medium. Thus a cross section of the emerging beam can be viewed as a planar, secondary, quasi-homogeneous source. Consider now the field generated by such a source in the half-space $z > L$. The radiant intensity $\mathcal{J}^{(L)}$ of a field generated by a quasi-homogeneous source is given by the formula Ref. 16, Sect. 5.3.2)

$$\mathcal{J}^{(L)}(k\mathbf{s}_\perp) = (2\pi k)^2 \tilde{I}(0, L)\tilde{\mu}(k\mathbf{s}_\perp, L)\cos^2\theta\,, \tag{12}$$

where $\tilde{\mu}(\mathbf{f}, L)$ and $\tilde{I}(\mathbf{f}, L)$ are the Fourier transforms of the spectral degree of coherence and of the intensity of the field in the plane $z = L$, respectively, $\mathbf{s}_\perp$ is a vectorial projection of the three-dimensional unit outward vector $\hat{\mathbf{s}}$ onto the source plane (see Fig. 1), and $|\mathbf{s}_\perp| = \sin\theta$. Since the field produced by the secondary source is assumed to propagate only close to the z axis, one can make the approximation $\sin\theta \approx \theta$, $\cos\theta \approx 1$. Under these conditions, one obtains upon taking the inverse Fourier transform in Eq. (12), the following approximate expression for the spectral degree of coherence of the source in terms of the radiant intensity[19]:

$$\mu(\rho, L) = \frac{1}{(2\pi k)^2 \tilde{I}(0, L)} \int_0^\infty \mathrm{d}s_\perp\, s_\perp J_0(ks_\perp\rho)\mathcal{J}^{(L)}(ks_\perp)\,. \tag{13}$$

Here we have made use of the isotropy of the spectral degree of coherence of the source and performed the angular integration.

Similarly, one can determine the spectral degree of coherence of the incident beam by measuring its radiant intensity $\mathcal{J}^{(0)}$. The spectral degree of coherence can then be expressed as

$$\mu_0(\rho) = \frac{1}{(2\pi k)^2 \tilde{I}_0(0)} \int_0^\infty \mathrm{d}s_\perp\, s_\perp J_0(ks_\perp\rho)\mathcal{J}^{(0)}(ks_\perp)\,, \tag{14}$$

where, in the absence of absorption in the medium, $\tilde{I}_0(0) = \tilde{I}(0, L)$.[20] On combining Eqs. (13) and (14), it follows that

$$\frac{\mu_0(\rho)}{\mu(\rho, L)} = \frac{\int_0^\infty \mathrm{d}s_\perp\, s_\perp J_0(ks_\perp\rho)\mathcal{J}^{(0)}(ks_\perp)}{\int_0^\infty \mathrm{d}s_\perp\, s_\perp J_0(ks_\perp\rho)\mathcal{J}^{(L)}(ks_\perp)}\,. \tag{15}$$

Equations (11) and (15) represent the solution to the problem of determining the spectral density of dielectric constant fluctuations in a random medium under strong fluctuation conditions.

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15. It is a good approximation provided that the dielectric constant fluctuations are sufficiently small, $|\langle\epsilon_1^2(\rho, z)\rangle| \ll \langle\epsilon(\rho, z)\rangle^2$. We note, however, that this requirement does not impose any restriction on the magnitude of the fluctuations of optical fields (Ref. 13, Chap. 3).
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19. This approximation amounts to neglecting high spatial Fourier components associated with evanescent waves.
20. This assertion follows from the conservation of energy in the absence of absorption, which, in the paraxial approximation, may be expressed by the relation $\int \mathrm{d}^2\rho I_0(\rho) = \int \mathrm{d}^2\rho I(\rho, L)$.